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ISOVARIANT BORSUK-ULAM TYPE RESULTS AND THEIR CONVERSE

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0. The Borsuk-Ulam theorem

In this note, we first make a brief survey of Borsuk-Ulam type theorems, and next introduce some results on the isovariant Borsuk-Ulam theorem and its converse from [22, 23].

K. Borsuk (1905–82) showed the following three results in 1933.

Theorem 0.1 ([21]).

(B1) *If $f : S^n \to S^n$ is antipodal, i.e., $f(-x) = -f(x)$ for all $x \in S^n$, then f is essential, i.e., f is not null-homotopic.*

(B2) *For any continuous map $f : S^n \to \mathbb{R}^n$, there exists $x_0 \in S^n$ such that $f(x_0) = f(-x_0)$.*

(B3) *Suppose $S^n = \bigcup_{i=0}^{n} F_i$, F_i: nonempty closed sets. Then some F_i contains an antipodal pair; $\{x_0, -x_0\} \subset F_i$. (Lusternik-Schnirelmann 1930)*

The second result was conjectured by S. Ulam; so it is usually called the Borsuk-Ulam theorem. It is known that the Borsuk-Ulam theorem has various equivalent statements; indeed, the above statements (B1)–(B3) are equivalent, and in addition, the following statements are also equivalent to the Borsuk-Ulam theorem.

(B4) If $f : S^n \to \mathbb{R}^n$ is antipodal, then $f^{-1}(0) \neq \emptyset$.

(B5) If $f : S^n \to S^m$ is antipodal, then $n \leq m$.

0.1. **Generalization.** Each of (B1) – (B5) has various generalizations and related topics. Indeed (B1) says that the degree of f is nonzero; in fact, it is well known that $\deg f$ is odd. Thus (B1) is related to the degree of (equivariant) maps or degree theory. Recently Hara [11] and Inoue [13] obtained a natural extension of (B1) for equivariant maps between Stiefel manifolds with standard $O(n)$- or $\mathbb{Z}_p^k$-action.

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Statements (B2) and (B4) are related to coincidence theory or fixed point theory, and there are various researches in this field; see, for example, Gonçalves-Jaworowski-Pergher [8], Gonçalves et al. [9], Gonçalves-Wong [10].

Statement (B3) is related to the Lusternik-Schnirelmann category or Lusternik-Schnirelmann theory, which provides lower estimate for the number of critical points of a smooth function. For example, (B3) implies $\operatorname{cat} \mathbb{R}P^n \geq n$ and so we obtain $\operatorname{cat} \mathbb{R}P^n = n$, where $\operatorname{cat} X$ denotes the Lusternik-Schnirelmann category of X, i.e., $\operatorname{cat} X := \min\{n | X = \bigcup_{i=0}^n F_i$, each F_i is closed and contractible in $X\}$.

0.2. **Equivariant generalization.** From the viewpoint of transformation groups, (B5) can be rephrased as follows: If there is a $\mathbb{Z}_2$-map $f : S^n \to S^m$, then $n \leq m$ holds, where $\mathbb{Z}_2$ acts antipodally on the spheres. This formulation has a lot of equivariant generalizations; see, for example, Jaworowski [14], Dold [6], Fadell-Husseini [7], Marzantowicz [18], Bartsch [1], Komiya [16], Hara-Minami [12], etc. We recall some well-known equivariant generalizations. A direct generalization of (B5) is the following.

Theorem 0.2. *Suppose that $G \neq 1$ acts freely on S^n, S^m. If there is a G-map $f : S^n \to S^m$, then $n \leq m$ holds. (Dold [6], Kobayashi [15], Laitinen [17] etc.)*

The proof of Theorem 0.2 is reduced to the case $G = \mathbb{Z}_p$. An important fact is that the degree of a self G-map $f : S^n \to S^n$ is nonzero; in fact $\deg f \equiv 1 \mod p$.

Remark. This result still holds for free finite G-CW complexes homotopy equivalent to spheres.

In nonfree case, the following is known.

Theorem 0.3. *If there is a $\mathbb{Z}_p^k$-map (or T^k-map) $f : S^n \to S^m$, where $\mathbb{Z}_p^k$ or T^k acts fixed-point-freely on spheres, then $n \leq m$ holds. (Fadell-Husseini [7], Marzantowicz [18], etc.) Moreover this result still holds for $\mathbb{Z}_p$ (or $\mathbb{Q}$)-homology spheres. (Clapp-Puppe [4].)*

A euclidean space V with linear G-action is called a G-representation. We may suppose that the action is orthogonal. Let SV denote the unit sphere of a G-representation V. In this case, we say that G acts linearly on SV or that SV is a linear G-sphere.

A fundamental question is: For which finite groups does a Borsuk-Ulam type result hold? T. Bartsch [1] answered this question as follows.

Theorem 0.4 ([1]). *Suppose that G is a finite group. The "weak" Borsuk-Ulam theorem for linear G-spheres holds if and only if G is a p-group. Namely G has the following property (W) if and only if G is a p-group.*

(W) : *There exists a monotonely increasing function φ_G diverging to infinity such that for any linear G-spheres SV, SW ($V^G = W^G = 0$) with a G-map $f : SV \to SW$, the inequality $\varphi_G(\dim SV) \leq \dim SW$ holds.*

By Theorem 0.3, one can take the identity map as φ_G for $G = \mathbb{Z}_p^k$, which is the best possible function satisfying (W); such a function φ_G is called the Borsuk-Ulam function. In general, it is difficult to determine the Borsuk-Ulam function, but a few results are known; see [1] for relevant results.

For other topics on the Borsuk-Ulam theorem, see also Steinlein [25, 26], Matoušek [19].

1. The isovariant Borsuk-Ulam theorem

Let G be a compact Lie group. Let X, Y be G-spaces, and V, W G-representations.

Definition 1. A continuous map $f : X \to Y$ is called *G-isovariant* (or isovariant) if f is G-equivariant and preserves the isotropy groups, i.e., $G_{f(x)} = G_x$ for any $x \in X$.

A. G. Wasserman [27] first studied an isovariant version of the Borsuk-Ulam theorem. Using the Borsuk-Ulam theorem for free $\mathbb{Z}_p$-actions, one can obtain the following result.

Theorem 1.1 (Isovariant Borsuk-Ulam theorem). *Let G be a solvable compact Lie group. If there is an isovariant map $f : SV \to SW$, then*

$$\dim SV - \dim SV^G \leq \dim SW - \dim SW^G.$$

We note that this result still holds for semilinear actions on spheres.

Definition 2. The smooth G-action on a (homotopy) sphere M is called *semilinear* if for any $H \leq G$, M^H is a (homotopy) sphere or $\emptyset$. We call such a G-manifold M a *semilinear G-sphere.*

Theorem 1.2 ([21]). *Let G be a solvable compact Lie group and let M, N be semilinear G-spheres. If there is an isovariant map $f : M \to N$, then*

$$\dim M - \dim M^G \leq \dim N - \dim N^G.$$

It is still open whether Theorem 1.1 holds for an arbitrary compact Lie group, but Theorem 1.2 does not hold if G is nonsolvable.

Theorem 1.3 ([21]). *Let G be a nonsolvable compact Lie group. There are fixed-point-free semilinear G-spheres M_n, $n \geq 1$, with $\lim_{n\to\infty} \dim M_n = \infty$ and a representation sphere SW such that there is an isovariant maps $f_n : M_n \to SW$ for every n.*

Consequently, we obtain a Bartsch type result for semilinear actions; namely, the isovariant Borsuk-Ulam theorem for semilinear G-spheres holds if and only if G is solvable.

Remark. Bartsch's result, Theorem 0.4, still holds for semilinear G-spheres.

2. The converse of the isovariant Borsuk-Ulam theorem

Let G be a solvable compact Lie group. A subgroup means a closed subgroup. As mentioned in the previous section, the isovariant Borsuk-Ulam theorem holds for G. We would like to consider the converse.

If there is an isovariant map $f : SV \to SW$, then $f^H : SV^H \to SW^H$, $H \lhd K \leq G$, is K/H-isovariant. Since K/H is also solvable, we can apply the isovariant Borsuk-Ulam theorem to f^H. Hence we have

Proposition 2.1. *Let G be a solvable compact Lie group. If there is an isovariant map $f : SV \to SW$, then*

$(C_{V,W})$: $\dim SV^H - \dim SV^K \leq \dim SW^H - \dim SW^K$ *for any pair of closed subgroups $H \lhd K$.*

We formulate the converse problem of the isovariant Borsuk-Ulam theorem as follows.

Question. Let G be a solvable compact Lie group. Suppose that a pair (V, W) of G-representations satisfies

(a) $\operatorname{Iso} SV \subset \operatorname{Iso} SW$,

(b) $(C_{V,W})$.

Is there a G-isovariant map $f : SV \to SW$ (or $f : V \to W$)?

Remark. (1): The condition (a) is obviously necessary. However if G is abelian, then one can see that the condition (b) implies (a); so the condition (a) can be omitted.

(2) Note that there exists an isovariant map $f : SV \to SW$ if and only if there exists an isovariant map $f : V \to W$.

Definition 3. If this question is affirmative for G, we say that G has the *complete Borsuk-Ulam property* (or G is a *complete Borsuk-Ulam group*).

Unfortunately the complete answer is not known yet, but there are some partial results. In this note, we would like to give the outline of proof of the following theorem; the full detail will appear in [23].

Theorem 2.2. *The following groups have the complete Borsuk-Ulam property.*

(1) *finite abelian p-group,*

(2) $\mathbb{Z}_{p^n q^m}$,
(3) $\mathbb{Z}_{pqr}$,

where p, q, r are prime numbers.

Let T_k, $k \in \mathbb{Z}$, be the irreducible S^1-representation given by $t \cdot z := t^k z$, $t \in S^1$ ($\subset \mathbb{C}$), $z \in T_k$ ($= \mathbb{C}$). Restricting T_k to $\mathbb{Z}_n \subset S^1$, we have a $\mathbb{Z}_n$-representation, denoted by the same symbol T_k. For simplicity we here treat only complex representations.

2.1. **Proof of Theorem 2.2 (1) (outline).** Let us consider the case $G = \mathbb{Z}_p$. Then T_k, $0 \le k \le p-1$, are all irreducible $\mathbb{Z}_p$-representations. We may suppose $V^G = W^G = 0$. In fact, one can see that there exists an isovariant map $f : V \to W$ if and only if there exists an isovariant map $f : V_G \to W_G$, where V_G denotes the orthogonal complement of V^G in V. Therefore we may set $V = T_{k_1} \oplus \cdots \oplus T_{k_n}$, $W = T_{l_1} \oplus \cdots \oplus T_{l_m}$, where k_i, l_i are prime to $|G|$.

An isovariant map $f : T_k \to T_l$ is defined by $f_{k,l}(z) = \xi^{k'l} z$, where $k'k \equiv 1 \mod |G|$. Since condition $(C_{V,W})$ implies $n \le m$, one can construct an isovariant map $f : V \to W$ using $f_{k,l}$.

For a general abelian p-group, a similar argument shows Theorem 2.2 (1).

2.2. **Proof of Theorem 2.2 (2) (outline).**

Definition 4. A pair of representations (V, W) is called *primitive* if V and W cannot be decomposed into $V = V_1 \oplus V_2$, $W = W_1 \oplus W_2$ such that $(V_i, W_i) \ne (0,0)$ satisfies (C_{V_i,W_i}), $i = 1, 2$.

If there are isovariant maps $f_i : V_i \to W_i$, then $f_1 \oplus f_2 : V_1 \oplus W_1 \to V_2 \oplus W_2$ is also isovariant; therefore it suffices to construct an isovariant map between each primitive pair.

Let us consider $G = \mathbb{Z}_{pq}$ for example. Clearly $(0, T_s)$ and (T_k, T_l), $(k, |G|) = (l, |G|)$, are primitive, and one can easily construct isovariant maps between these representations as in the proof of (1). In addition, a new primitive pair $(T_1, T_p \oplus T_q)$ appears for $G = \mathbb{Z}_{pq}$. In this case an isovariant map exists; for example, the map defined by $f : z \mapsto (z^p, z^q)$ is isovariant. These pairs mentioned above are essentially all primitive pairs for $\mathbb{Z}_{pq}$. Therefore $\mathbb{Z}_{pq}$ has the complete isovariant Borsuk-Ulam property.

For $\mathbb{Z}_{p^n q^m}$, other primitive pairs appear, but one can directly define isovariant maps in a similar way. For example, $(T_p \oplus T_q,\ T_{p^2} \oplus T_{pq} \oplus T_{q^2})$ is primitive for $\mathbb{Z}_{p^n q^m}$, $n, m \ge 2$. In this case there is an isovariant map; for example $f : (z_1, z_2) \mapsto (z_1^p, z_1^q + z_2^p, z_2^q)$ is isovariant. Thus one can see that $\mathbb{Z}_{p^n q^m}$ has the complete isovariant Borsuk-Ulam property.

2.3. **Proof of Theorem 2.2 (3) (outline).** Next consider the case of $\mathbb{Z}_{pqr}$. The proof is more complicated.

For all primitive pairs except one type, one can directly define isovariant maps as before. The exception is the following type of primitive pair:

$$(T_p \oplus T_q \oplus T_r,\ T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}).$$

If there is an isovariant map for this pair, it turns out that $\mathbb{Z}_{pqr}$ has the complete isovariant Borsuk-Ulam property. It seems, however, difficult to directly define an isovariant map; so we would like to use equivariant obstruction theory.

The question is the following:

Question. Is there a $\mathbb{Z}_{pqr}$-isovariant map

$$f : T_p \oplus T_q \oplus T_r \to T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}?$$

The answer is yes. Actually we shall show the existence of an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{pr}).$$

Therefore we see that $\mathbb{Z}_{pqr}$ has the complete Borsuk-Ulam property.

3. The existence of an isovariant map

We shall discus the above question in a more general setting. Let $G = S^1$ and let M be a rational homology sphere with *pseudofree* S^1-action.

Definition 5 (Montgomery-Yang). An S^1-action on M is *pseudofree* if

(1) the action is effective, and
(2) the singular set $M^{>1} := \bigcup_{1 \neq H \leq S^1} M^H$ consists of finitely many exceptional orbits.

Here an orbit $G(x)$ is called exceptional if $G(x) \cong S^1/C$, $(1 \neq C < S^1)$.

Example 3.1. Let $V = T_p \oplus T_q \oplus T_r$. Then the S^1-action on SV is pseudofree. Indeed it is clearly effective, and

$$\begin{aligned} SV^{>1} &= ST_p \coprod ST_q \coprod ST_r \\ &\cong S^1/\mathbb{Z}_p \coprod S^1/\mathbb{Z}_q \coprod S^1/\mathbb{Z}_r \end{aligned}$$

Remark. There are many "exotic" pseudofree S^1-actions on high-dimensional homotopy spheres. (Montgomery-Yang [20], Petrie [24].)

Let SW be any S^1-representation sphere. We consider an S^1-isovariant map $f : M \to SW$.

The result is the following:

Theorem 3.2. *With the above notation, there is an S^1-isovariant map $f : M \to SW$ if and only if*

(I): $\operatorname{Iso} M \subset \operatorname{Iso} SW$,
(PF1): $\dim M - 1 \leq \dim SW - \dim SW^H$ *when* $1 \neq H \leq C$ *for some* $C \in \operatorname{Iso} M$,
(PF2): $\dim M + 1 \leq \dim SW - \dim SW^H$ *when* $1 \neq H \not\leq C$ *for every* $C \in \operatorname{Iso} M$.

3.1. **Examples.** We give some examples. Let p, q, r be pairwise coprime integers greater than 1.

Example 3.3. There is an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. (PF1) and (PF2) are fulfilled. One can see $\operatorname{Iso} M = \{1, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r\}$ and $\operatorname{Iso} SW = \{1, \mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r, \mathbb{Z}_{pq}, \mathbb{Z}_{qr}, \mathbb{Z}_{rp}\}$; hence $\operatorname{Iso} M \subset \operatorname{Iso} SW$. □

Example 3.4. There is not an S^1-isovariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

Proof. (PF1) is not fulfilled. □

Remark. There is an S^1-equivariant map

$$f : S(T_p \oplus T_q \oplus T_r) \to S(T_{pq} \oplus T_{qr} \oplus T_{rp}).$$

By Example 3.3, we see that $\mathbb{Z}_{pqr}$ has the complete Borsuk-Ulam property.

3.2. **Proof of Theorem 3.2 (outline).** We shall give the outline of Theorem 3.2. The full detail will appear in [22]. Set $Y := SW \setminus SW^{>1}$. Note that S^1 acts freely on Y. Let N_i be an S^1-tubular neighborhood of each exceptional orbit in M. By the slice theorem, N_i is identified with $S^1 \times_{C_i} DU_i$ $(1 \leq i \leq r)$, where C_i is the isotropy group of the exceptional orbit and U_i is the slice C_i-representation. Set $X := M \setminus (\coprod_i \operatorname{int} N_i)$. Note that S^1 acts freely on X.

The "only if" part is proved by the (isovariant) Borsuk-Ulam theorem. Indeed we can show (PF1) as follows. Take a point $x \in M$ with $G_x = C$ and a C-invariant closed neighborhood B of x C-diffeomorphic to some unit disk DV. Hence we obtain an H-isovariant map $f : SV \to SW$. Applying the isovariant Borsuk-Ulam theorem to f, we have (PF1).

We next show (PF2). Since f is isovariant, f maps M into $SW \setminus SW^H$, and since $SW \setminus SW^H$ is S^1-homotopy equivalent to SW_H, we obtain an S^1-map $g : M \to SW_H$. By Theorem 0.3, we obtain (PF2).

To show the converse, we begin with the following lemma.

Lemma 3.5. *There is an S^1-isovariant map $\tilde{f}_i : N_i \to SW$.*

Proof. Let $N_i = N = S^1 \times_C DV$, where C is the isotropy group of the exceptional orbit and V is the slice representation. Similarly take a closed S^1-tubular neighborhood $N'$of an exceptional orbit with isotropy group C, and set $N' = S^1 \times_C DV'$. By (PF1), we see that $\dim SV + 1 \leq \dim SV' - \dim SV'^{>1}$. Since C acts freely on SV, by obstruction theory, there is an C-map $g : SV \to SV' \setminus SV^{>1} \subset SW$, and so we obtain a C-isovariant map $g : SV \to SW$. Taking a cone, we have a C-isovariant map $\tilde{g} : DV \to DV'$; hence there is an S^1-isovariant map $\tilde{f} = S^1 \times_C \tilde{g} : N \to N' \subset SW$. □

Set $f_i := \tilde{f}_i|_{\partial N_i} : \partial N_i \to Y$, and $f := \coprod_i f_i : \partial X \to Y$. If f is extended to an S^1-map $F : X \to Y$, by gluing the maps, we obtain an S^1-isovariant map

$$F \cup (\coprod_i \tilde{f}_i) : M \to SW.$$

Thus it suffices to investigate the following question:

(Q) Is there an extension $F : X \to Y$ of $f : \partial X \to Y$?

Since S^1 acts freely on X and Y, the obstruction to an extension lies in

$$H^*(X/S^1, \partial X/S^1; \pi_{*-1}(Y)).$$

Set $k = \dim SW - \dim SW^{>1}$. A standard computation shows

Lemma 3.6. (1) *Y is $(k-2)$-connected and $(k-1)$-simple.*

(2) *$\pi_{k-1}(Y) \cong H_{k-1}(Y) \cong \oplus_{H \in \mathcal{A}} \mathbb{Z}$, where $\mathcal{A} := \{H \in \operatorname{Iso} SW | \dim SW^H = \dim SW^{>1}\}$, and generators are represented by SW_H, $H \in \mathcal{A}$.*

Note that $\dim M - 1 \leq k$ by (PF1) and (PF2). We divide into two cases.

Case I: $\dim M - 1 < k$ (i.e., $\dim X/S^1 < k$). In this case, we see that

$$H^*(X/S^1, \partial X/S^1; \pi_{*-1}(Y)) = 0$$

by dimensional reason. Hence the obstruction vanishes and there exists an extension $F : X \to Y$.

Case II: $\dim M - 1 = k$ (i.e., $\dim X/S^1 = k$). The obstruction $\gamma_{S^1}(f)$ to an extension lies in

$$H^k(X/S^1, \partial X/S^1; \pi_{k-1}(Y)) \cong \oplus_{H \in \mathcal{A}} \mathbb{Z}.$$
$$(H^l(X/S^1, \partial X/S^1; \pi_{k-1}(Y)) = 0, l \neq k)$$

To detect the obstruction, we introduce the multidegree.

3.3. **Multidegree.** Let $N = S^1 \times_C DU \subset M$, $1 \neq C \in \operatorname{Iso}(M)$, $\dim M - 1 = \dim U = k$, and $f : \partial N \to Y$: S^1-map, $\bar{f} = f|_{SU} : SU \to Y$: C-map.

Definition 6. $\operatorname{Deg} f := \bar{f}_*([SU]) \in \oplus_{H \in \mathcal{A}} \mathbb{Z}$, $\bar{f}_* : H_{k-1}(SU) \to H_{k-1}(Y)$, under identifying $H_{k-1}(Y)$ with $\oplus_{H \in \mathcal{A}} \mathbb{Z}$.

Then the obstruction $\gamma_{S^1}(f)$ is described by the multidegrees.

Proposition 3.7. *Let $F_0 : X \to Y$ be a fixed S^1-map (not necessary extending f). Set $f_{0,i} = F_0|_{\partial N_i}$. Then*

$$\gamma_{S^1}(f) = \sum_{i=1}^{r} (\operatorname{Deg} f_i - \operatorname{Deg} f_{0,i})/|C_i|.$$

Remark. (1) There always exists F_0 .

(2) $\operatorname{Deg} f_i - \operatorname{Deg} f_{0,i} \in \oplus_{H\in\mathcal{A}}|C_i|\mathbb{Z}$ by the equivariant Hopf type result. (See the next section.)

Using this proposition and equivariant Hopf type results in the next section, we can choose S^1-isovariant maps $\tilde{f}_i : N_i \to SW$ so that $\gamma_{S^1}(f) = 0$.

4. Equivariant Hopf type results

Let $N = S^1 \times_C DU$ ($\subset M$), $\dim M - 1 = k$ as before. Then the following Hopf type theorem holds.

Theorem 4.1 ([22]). (1) $\operatorname{Deg} : [\partial N, Y]_{S^1} \to \oplus_{H\in\mathcal{A}}\mathbb{Z}$ *is injective.*

(2) *The image of* $\operatorname{Deg} - \operatorname{Deg} f_0$ *coincides with* $\oplus_{H\in\mathcal{A}}|C|\mathbb{Z}$*, where* f_0 *is any fixed* S^1*-map.*

The next result shows the extendability of $f : \partial N = S^1 \times_C SU \to Y$. Set $\operatorname{Deg} f = (d_H(f))_{H\in\mathcal{A}} \in \oplus_{H\in\mathcal{A}}\mathbb{Z}$.

Theorem 4.2 ([22]). (1) $f : \partial N \to Y$ *is extendable to an* S^1*-isovariant map* $\tilde{f} : N \to SW$ *if and only if* $d_H(f) = 0$ *for any* $H \in \mathcal{A}$ *with* $H \not\leq C$.

(2) *For any extendable* f *and for any* $(a_H) \in \oplus_{H\in\mathcal{A}}|C|\mathbb{Z}$ *satisfying* $a_H = 0$ *for* $H \in \mathcal{A}$ *with* $H \not\leq C$*, there exists an* S^1*-map* $f' : \partial N \to Y$ *such that* f' *is extendable to an* S^1*-isovariant map* $\tilde{f}' : N \to SW$ *and* $\operatorname{Deg} f' = \operatorname{Deg} f + (a_H)$.

4.1. **Example of multidegrees.** Finally we give some examples. Take S^1-representations $V = T_q \oplus T_q \oplus T_r$ and $W = T_1 \oplus T_{pq} \oplus T_{qr} \oplus T_{rp}$, where p, q, r are distinct primes. Let us consider linear spheres SV, SW. Let N_i be a closed S^1-tubular neighborhood of the exceptional orbit $ST_i \cong S^1/\mathbb{Z}_i$ in SV, where $i = p$, q, r, and then N_i is identified with $ST_i \times D(T_j \oplus T_k) \cong S^1 \times_{\mathbb{Z}_i} D(T_j \oplus T_k)$. Thus we may set

$$N_p = \{(z_1, z_2, z_3) \in V \mid |z_1| = 1, \|(z_2, z_3)\| \leq 1\},$$
$$N_q = \{(z_1, z_2, z_3) \in V \mid |z_2| = 1, \|(z_1, z_3)\| \leq 1\},$$
$$N_r = \{(z_1, z_2, z_3) \in V \mid |z_3| = 1, \|(z_1, z_2)\| \leq 1\}.$$

We have $\mathcal{A} = \{\mathbb{Z}_p, \mathbb{Z}_q, \mathbb{Z}_r\}$; hence we can set

$$\operatorname{Deg} f = (d_{\mathbb{Z}_p}(f), d_{\mathbb{Z}_q}(f), d_{\mathbb{Z}_r}(f)) \in \mathbb{Z}^3.$$

Take positive integers α, β, γ, δ, ξ, η such that $\alpha p - \beta q = 1$, $\gamma q - \delta r = 1$, $\xi r - \eta p = 1$.

Example 4.3. We define $g_i : V \to W$ as follows:

$$g_p(z_1, z_2, z_3) = (z_3^{\xi}\bar{z}_1^{\eta}, z_1^q, z_2^r, z_1^r),$$
$$g_q(z_1, z_2, z_3) = (z_1^{\alpha}\bar{z}_2^{\beta}, z_2^p, z_2^r, z_3^p),$$
$$g_r(z_1, z_2, z_3) = (z_2^{\gamma}\bar{z}_3^{\delta}, z_1^q, z_3^q, z_3^p).$$

Restricting g_i to N_i, we obtain an S^1-map $h_i := g_{i|} : N_i \to W$. Since $h_i^{-1}(0) = \emptyset$, we have an S^1-map $\tilde{f}_i := h_i/\|h_i\| : N_i \to SW$. Moreover $\tilde{f}_i$ is an S^1-isovariant. Set $f_i = \tilde{f}_{i|\partial N_i}$. Then $d_{\mathbb{Z}_p}(f_p)$ is equal to the degree of the map $f_p' : S(T_q \oplus T_r) \to S(T_1 \oplus T_{qr})$;

$$(z_2, z_3) \mapsto (z_3^{\xi}\bar{z}_1^{\eta}, z_2^r)/\|(z_3^{\xi}\bar{z}_1^{\eta}, z_2^r)\|,$$

where z_1 is any fixed nonzero number. Hence we have $d_{\mathbb{Z}_p}(f_p) = \xi r = 1 + \eta p$. Similarly one can see that $d_{\mathbb{Z}_q}(f_p) = d_{\mathbb{Z}_r}(f_p) = 0$. Thus we obtain

$$\operatorname{Deg} f_p = (1 + \eta p, 0, 0).$$

In a similar way, we have

$$\operatorname{Deg} f_q = (0, 1 + \beta q, 0),$$
$$\operatorname{Deg} f_r = (0, 0, 1 + \delta r).$$

Example 4.4. Next we consider the following S^1-maps $g_i' : V \to W$:

$$g_p'(z_1, z_2, z_3) = (z_2^{\gamma}\bar{z}_3^{\delta}, z_1^q, z_2^r + z_3^q, z_1^r),$$
$$g_q'(z_1, z_2, z_3) = (z_3^{\xi}\bar{z}_1^{\eta}, z_2^p, z_2^r, z_3^p + z_1^r),$$
$$g_r'(z_1, z_2, z_3) = (z_1^{\alpha}\bar{z}_2^{\beta}, z_1^q + z_2^p, z_3^q, z_3^p).$$

Then by restriction and normalization, we obtain S^1-isovariant maps $\tilde{f}_i' : N_i \to SW$ and $f_i' : \partial N_i \to SW$, respectively. In this case, one can see that

$$\operatorname{Deg} f_p' = (1, 0, 0),$$
$$\operatorname{Deg} f_q' = (0, 1, 0),$$
$$\operatorname{Deg} f_r' = (0, 0, 1).$$

In fact, for example, $d_{\mathbb{Z}_p}(f_p') = 1$ is showed as follows. Consider the map $\psi : T_q \oplus T_r \setminus 0 \to T_1 \oplus T_{qr} \setminus 0$; $(z_2, z_3) \mapsto (z_2^{\gamma}\bar{z}_3^{\delta}, z_1^q, z_2^r + z_3^q)$. One can see that $\psi^{-1}(1, 0) = \{((-1)^{\delta}, (-1)^{\gamma})\}$ and the Jacobian is $\gamma q + r\delta > 0$; hence $(1, 0) \in T_1 \oplus T_{qr} \setminus 0$ is a regular value, and so $\deg \psi = 1$.

$Y = SW \setminus SW^{>1}$. Let $[\partial N_i, Y]_{S^1}^{\text{ext}}$, $i = p,\ q,\ r$, denote the set of S^1-homotopy classes of S^1-maps extended to S^1-isovariant maps from N_i to SW. By Theorems 4.1 and 4.2, we see the following.

Proposition 4.5. *The map $D_i : [\partial N_i, Y]_{S^1}^{\text{ext}} \to \mathbb{Z}, [f] \mapsto (d_{\mathbb{Z}_i}(f) - 1)/i$, is a bijection for $i = p,\ q,\ r$.*

For the above maps, we have $D_p(f_p) = \eta$ and $D_p(f'_p) = 0$.

Example 4.6. We next define another S^1-map $f_{0,i}$ as follows. Define an S^1-map $g_0 : V \to W$ by setting

$$g_0(z_1, z_2, z_3) = (z_1^\alpha \bar{z}_2^\beta + z_2^\gamma \bar{z}_3^\delta + z_3^\xi \bar{z}_1^\eta, z_1^q, z_2^r, z_3^p).$$

Since g_0 maps the free part of V into the free part of W, by restriction and normalization, we have an S^1-map $f_{0,i} : \partial N_i \to Y$. In this case we have

$$\operatorname{Deg} f_{0,p} = (1 + \eta p, -\beta p, 0),$$

$$\operatorname{Deg} f_{0,q} = (0, 1 + \beta q, -\delta q),$$

$$\operatorname{Deg} f_{0,r} = (-\eta r, 0, 1 + \delta r).$$

By Theorem 4.2, each $f_{0,i}$ cannot be isovariantly extended on N_i.

However, restricting g_0 on $X = SV \setminus \operatorname{int}(N_p \cup N_q \cup N_r)$, one can regard g_0 as an S^1-map from X to Y. Consequently it turns out that $\coprod_i f_{0,i}$ can be extended on X. Consider the S^1-maps $f = \coprod_i f_i : \partial N_i \to Y$ and $f' = \coprod_i f'_i : \partial N_i \to Y$ in Examples 2 and 3. By Proposition 3.7, the obstruction $\gamma_{S^1}(f)$ to an extension on X is described as $\gamma_{S^1}(f) = (\eta, \beta, \delta)$ and $\gamma_{S^1}(f') = (0, 0, 0)$; hence f cannot be extended on X, but f' can.

We also note the following.

Proposition 4.7. *An S^1-isovariant map $\tilde{h} = \coprod_i \tilde{h}_p : \coprod_i N_i \to SW$ is isovariantly extended on SV if and only if* $\operatorname{Deg} h_p = (1, 0, 0)$, $\operatorname{Deg} h_q = (0, 1, 0)$ *and* $\operatorname{Deg} h_r = (0, 0, 1)$, *where $h_i = \tilde{h}_{i|\partial N_i}$.*

Proof. One can set $\operatorname{Deg} h_p = (1 + np, 0, 0)$, $\operatorname{Deg} h_q = (0, 1 + mq, 0)$ and $\operatorname{Deg} h_r = (0, 0, 1 + lr)$. Then one can see $\gamma_{S^1}(h) = (n, m, l)$, and so $\gamma_{S^1}(h) = 0$ if and only if $(n, m, l) = (0, 0, 0)$. $\square$

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